Towards Mechanising Probabilistic Properties of a Blockchain

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Abstract
We present our progress on the formalisation and mechanisation of a probabilistic model of a blockchain consensus protocol in Coq, taking steps towards the formal verification of its security properties, stated in terms of probabilities, in an adversarial environment.

1 Introduction
Blockchain consensus protocols are a family of open distributed byzantine [5] consensus protocols, where an arbitrary number of (potentially malicious) parties can participate at any point of time. Previous formalisation efforts considered a simple model of blockchain-based consensus and established its basic safety properties in Coq [8], yet they did not address any security properties, which inherently require to incorporate some notion of probability.

Fundamentally, a blockchain consensus protocol allows a set of independent actors communicating over an asynchronous network to maintain a shared public ledger, robust to adversarial attacks; this is achieved by using the calculation of a pre-image of a hash function as a validation tool [6]. Any adversary attempting to disrupt the consensus must first produce a sufficiently long validated chain, an action that becomes increasingly improbable as the length of the public chain increases. The work on Bitcoin Backbone Protocol (BBP) by Garay et al. [3] stated the Chain Growth and Common Prefix properties. For example, the former states that, given certain bounds on the ratio of adversarial parties, a consensus on the prefix of any chain held by an honest actor could be guaranteed to a high likelihood in a semi-synchronous setting. In this proposed talk, we will present our ongoing work on the formalisation of the BBP model and the proof of its security properties in Coq.

2 A Model for Bitcoin Backbone Protocol
2.1 State-Space of a Byzantine Distributed System
To capture the semantics of the BBP model, our formalisation must encode the following components of the global system state:

- **Network State.** Our definition simulates a $\Delta$-bounded-synchronous network (the network operation is modelled as occurring at the level of discrete rounds, such that all messages will by delivered after $\Delta$ rounds) through the use of a queue of fixed length $\Delta$ (i.e., a delivery queue). To encode this, all messages sent by hosts in the network during a round are stored in a message pool; at the end of the round this collection is then placed into the queue, removing the first entry and delivering all its messages sequentially.

- **Blockchains.** We represent a blockchain in the system as a sequence of blocks, where each block contains a sequence of transactions, the hash value $\mathbb{N}^{[0..\kappa]} < T$ of the prior block and an integer proof-of-work [2]. As in the BBP model, $T$ represents the globally fixed hashing difficulty and $\kappa$ represents the output size of the hash function.

- **Actors and Adversaries.** We represent the state of honest actors as a collection of their local blockchains, received messages and transactions. We encode an adversary as an opaque parameterised type, thereby preventing introspection into its state and allowing for arbitrary adversarial strategies. We associate a boolean value with the internal state of each actor to record whether they are honest or not. This simple dichotomy allows the model to represent varying numbers of malicious actors during the execution, allowing the adversaries to corrupt honest actors.

- **Oracle state.** Oracles are a standard way of representing non-determinism within Coq [8]; we utilise an oracle to capture the non-deterministic nature of the hashing operation, whereby the hashing of an unseen block returns a random value. To ensure that the oracle produces consistent results (i.e., hashing the same block twice produces the same output) we represent the state of the oracle as a map between blocks and $\mathbb{N}^{[0..2^k]}$—an integer representing a $\kappa$-length hash result.

We encapsulate all these components of the state-space within the world data type, representing the entire history of the protocol execution until a certain round. As the state of the oracle is tied to the world state, representing the semantics of the system as a binary relation on worlds would prevent a probabilistic analysis of the random values returned by the oracle, we must instead define the semantics as a probabilistic computation.

2.2 Modelling Randomised System Executions
We encode the protocol semantics as the following step function:

$$\text{world} \_\text{step} : \text{World} \rightarrow \text{seq} \text{RndGen} \rightarrow \text{Comp} (\text{option World})$$

The first parameter is the initial world to start an execution from. The second parameter acts as a schedule [8], representing by the sequence of events and internal choices (RndGen) leading to the overall execution result. The system events from seq RndGen include generation of a transaction, the corruption of an honest actor, or a single call to a hash-function by an honest actor.

The world_step function, when provided an initial world and a schedule, iteratively consumes each event in the schedule and probabilistically updates the world state. The probabilistic result is represented by a value of type option to allow the execution to fail, as not all sequences of events are valid: for instance, it would not be valid for an honest actor to call the hash function if the adversary was active at that time. The crucial component of the world_step definition is its monadic return type Comp, representing the outcome of a randomised computation [9]. As embedded into Coq, expressions of type Comp define a domain-specific language for representing operations for generating and working with random bits [7]—precisely what we need to encode the random results of hashing used to generating transactions and blocks by actors.

Implemented in the style of the FCF library [7], randomised expressions (of type Comp A for some value type A, e.g., World) provide an ergonomic Haskell-style do-notation for constructing randomised computations. For example, the following code snippet shows a part of our definition of world_step in Coq that draws random hash values from the corresponding primitive hash, while randomly generating a new world with a freshly minted block:
2.3 Reasoning about Randomised Executions

As customary when reasoning about randomised algorithms, we state properties of our BBP executions in terms of probabilities. Since the results of an execution (Comp (option World)) represent the “syntax” of probabilistic distributions (dist A : A → ℝ[0, 1]), we can convert our computations into distributions of possible results via the probability monad defined by Affeldt and Hagiwara [1]:

bind : dist A → (A → dist B) → dist B
ret : A → dist A

eval_dist : Comp A → dist A

Now suppose we wish to reason about the probability that a given property F : option World → bool holds. We can represent that by stating that for all worlds reachable from some initial world w₀ via some schedule sc, by expressing the statement in terms of probabilities as follows:

∀ w₀, sc, eval_dist (world_step w₀ sc) w > 0 ⇒ F w

This can be reformulated via more transitional notation:

∀ w₀, sc, P [world_step w₀ sc] ⊢ F ] = 1, where

P [ a = b ] ≜ eval_dist a b
P [ a ] ≜ P [ a = true ]
fmap : Comp A → (A → B) → Comp B

3 Properties of the Bitcoin Backbone Protocol

Using constructions from Section 2, we state the security properties.

The first property, which asserts that there is an overall “progress” in the system with honest actors, relies on the auxiliary characteristic function X′ : ℕ → World → {0, 1}, such that X′ w returns 1 iff the round i was bounded successful in a world w, i.e., any honest actors were able to successfully mine a block during that round and all rounds from i − Δ to i had no successful mining attempts for the globally-fixed network delay Δ; otherwise, X′ w returns 0.

Definition 3.1 (Chain Growth Property). The property CGP : World → bool is defined with respect to a finite number of rounds Nrounds, fixed number of actors maxactors, and holds iff for a given world w, any round r ∈ [0, . . ., Nrounds], blockchain c, actor address addr ∈ {0, . . ., maxactors}, such that addr is honest in the world w and has c as its chain at round r, it is the case that for a “later” round s, such that s > r + Δ, and any other actor addr′ in the system, whose chain in s is c′, len c′ ≥ len c + \sum_{i∈[r..s−Δ]} X′_i w.

The second property defines a “preservation”-like notion similar to the classical consensus in a randomised blockchain-based setting.

Definition 3.2 (Common Prefix Property). The property CPP : World → bool is defined with respect to a consecutive sequence of rounds starting at i to j, a number of blocks k, and holds iff for a given world w, for any blockchains c₁, c₂, round r ∈ [i, . . ., j], such that chain c₁ is adopted by some actor¹ at round r, and c₂ is either adopted or diffused at round r, it is the case that pruning k blocks off the end of c₁ produces a prefix of c₂ and pruning k blocks off the end of c₂ produces a prefix of c₁.

Typical Executions A key innovation in the BBP proof is the choice to restrict the state of the model used to exclude exceptional situations, and instead consider the “average” case. Informally, this typical execution property can be described as follows:

Definition 3.3 (Typical Execution Property). The property TEP : seq RndGen → World → bool is defined wrt. a parameter ε : ℝ(0, 1) and holds iff for a schedule sc and a world w resulting from sc: (a) the number of bounded successful rounds and bounded uniquely successful rounds for the world are no more than an ε-ratio below their expected value given the schedule; (b) the number of successful rounds for the world is less than an ε-ratio above its expected value; (c) the number of blocks hashed by the adversary is less than an ε-ratio above its expected value.

In our development, the typical nature of the considered executions is encoded by the following assumption.

Assumption 3.1 (Typical Executions). ∀ sc : seq RndGen, ε : ℝ(0, 1),

P [world_step sc ⊢ TEP sc] = 1 − e^Ω(εk),

where Ω(ε) refers to any function that grows linearly in k.

3.1 Main Theorems

We state the main theorems in terms of the defined above properties. “Dotted” logical conjunction denotes its point-wise lifting to worlds.

Theorem 3.1 (Chain Growth Lemma). ∀ sc : seq RndGen,

P [world_step sc ⊢ CGP] = 1

Theorem 3.2 (Common Prefix Theorem). ∀ sc, k : ℕ, k > 2k \frac{Δ}{2ε},

P [world_step sc ⊢ (CPP_k ∧ TEP sc)] = 1

Care has been taken in the formulation of Theorem 3.2 to avoid incorporating the complex probabilities of the typical execution property into the theorem statement. Rather than proving the probability of a typical execution and the Common Prefix Property holding, we formulate the property as that given a typical execution, the Common Prefix Property holds with probability 1.

3.2 Elements of Our Mechanisation and Future Work

We built our mechanisation using the libraries by Affeldt and Hagiwara [1], which provides a probability framework implemented on top of the Ssreflect extension for Coq [4]. The mechanised versions of Definitions 3.1 and 3.2 are given as decidable predicates (i.e., returning bool), and require a careful choice of types of values they quantify over, which all must be finite. This, “Ssreflect-style” approach pays off by giving an access to a large library of rewriting lemmas for Σ-notation of sums, enabling very concise proofs (mostly, by rewriting) of classical probability properties.

Our proofs of Theorems 3.1 and 3.2 are by induction on the length of the schedule and are partially complete; most remaining efforts relate to verifying the message delivery mechanism. In the future, we are also planning to promote the statement of Assumption 3.1 to a Lemma, as its paper-and-pencil proof is given in the BBP paper [3].

¹We adopt the BBP terminology here, and define an actor as adopting a chain c at round r if the actor accepts chain c as its local chain during round r.
References