Mechanising Blockchain Consensus

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Abstract
We present the first formalisation of a blockchain-based distributed consensus protocol with a proof of its consistency mechanised in an interactive proof assistant.

Our development includes a reference mechanisation of the block forest data structure, necessary for implementing provably correct per-node protocol logic. We also define a model of a network, implementing the protocol in the form of a replicated state-transition system. The protocol’s executions are modeled via a small-step operational semantics for asynchronous message passing, in which packages can be rearranged or duplicated.

In this work, we focus on the notion of global system safety, proving a form of eventual consistency. To do so, we provide a library of theorems about a pure functional implementation of block forests, define an inductive system invariant, and show that, in a quiescent system state, it implies a global agreement on the state of per-node transaction ledgers. Our development is parametric w.r.t. implementations of several security primitives, such as hash-functions, a notion of a proof object, a Validator Acceptance Function, and a Fork Choice Rule. We precisely characterise the assumptions, made about these components for proving the global system consensus, and discuss their adequacy. All results described in this paper are formalised in the Coq proof assistant.

1 Introduction
The notion of decentralised blockchain-based consensus is a tremendous success of the modern science of distributed computing, made possible by the use of basic cryptography, and enabling many applications, including but not limited to cryptocurrencies, smart contracts, application-specific arbitration, voting, etc.

In a nutshell, the idea of a distributed consensus protocol based on blockchains, or transaction ledgers,\(^1\) is rather simple. In all such protocols, a number of stateful nodes (participants) are communicating with each other in an asynchronous message-passing style. In a message, a node (a) can announce a transaction, which typically represents a certain event in the system, depending on the previous state of the node or the entire network (we intentionally leave out the details of what can go into a transaction, as they are application-specific): a node can also (b) create and broadcast a block that contains the encoding of a certain vector of transactions, created locally or received via messages of type (a) from other nodes. Each recipient of a block message should then validate the block (i.e., check the consistency of the transaction sequence included in it), and, in some cases, append it to its local ledger, thus, extending its subjective view of the global sequence of transactions that have taken place in the system to date. The process continues as more messages are emitted and received.

In order to control the number of blocks in the system, distributed ledger protocols rely on certain cryptographic primitives, such as a hash-function hash defined both on transactions and blocks, a notion of a proof object necessary for defining the validity of a block, and an implementation of a Validator Acceptance Function (VAF) that is used to ensure that a block b is valid w.r.t. to a proof object pf. Having a block b and a proof object pf, one can check very fast whether VAF b pf is true or false. What appears to be difficult is to produce an instance of a proof object pf, as it requires computing a pre-image of the hash function with respect to the current state of the local ledger of a specific node. The exact specifics of designing a VAF and a discipline for minting blocks with VAF-valid proof objects, is a subject of active research, which is far beyond the scope of this paper, with the best known approaches being Proof-of-Work [8, 22] and Proof-of-Stake [2]. The computational hardness or probabilistic rarity of minting valid blocks is what controls the overall block population.

However, this setup by itself does not deliver a global consensus between the nodes. Indeed, in an asynchronous network, where messages can be rearranged, duplicated, or arbitrarily delayed, two different nodes \(n_1\) and \(n_2\) can receive different, or even conflicting, sets of valid blocks and decide to adopt them in their local ledgers. Assuming that initially all nodes share the same initial block (so-called Genesis Block), at any further state of the network, each two nodes’ ledgers can be in a fork relation, when neither of them is a prefix of the other. The consensus is enabled by fixing a globally known Fork Choice Rule (FCR) function, that provides a decidable strict total order on all possible chains of blocks and is transitive and irreflexive. Thus, upon receiving a block,
a node must check whether appending it to its local ledger is going to increase the ledger’s “weight”, and keep it if so, discarding it otherwise. Assuming every node follows the same FCR-imposed discipline for chain comparison, all participants will eventually share the same blockchain/transaction ledger instance.

Alas, the reality is a bit more complicated than the description above. For example, in a realistic fault-tolerant system implementation, nodes cannot afford to ignore blocks that arrive “out of order”, which is not uncommon in an asynchronous setting. Not registering such blocks in a node’s local state would pose serious liveness problems, as such nodes would be stuck with a “stale” local ledger, unable to progress along with the rest of the world. Furthermore, some nodes may not be active or known system-wide at the very beginning of communication, so they will start by first manifesting themselves, interacting only with a small set of peers they know. Finally, any node in the system should be able to request from its peers the set of publicly announced blocks these peers have witnessed in the past, so it would be possible for the node to “catch up” with the global state of the system, if, for instance, it has joined the network late or has been offline for some time.

In light of these and multiple other possible scenarios of distributed interaction, we believe that having a clean and principled model for rigorous formal reasoning about system-wide properties of distributed blockchain-based protocols is of paramount importance for gaining trust in the foundational principles of algorithms underlying, in particular, implementations of modern cryptocurrencies, such as Bitcoin [22], Ethereum [35] and Tezos [12].

In this work we provide such a model.

1.1 Our contributions
The aim of this work is to develop a formal model of a blockchain-based consensus protocol, along with a set of necessary reference data structures and a network semantics, with an agenda to formally study its properties, abstracting away the implementation details of security-related primitives. Our contributions towards this goal include the following formal artifacts:

- A description of a minimal set of security primitives: hash, VAF, FCR, along with a set of laws (axioms) they should abide, and a discussion of these laws’ adequacy wrt. real-world implementations;
- A reference implementation of block forests—a purely functional data structure implementing the local state of a node in the protocol in the presence of adopted out-of-order blocks, as well as a library of theorems about block forests, necessary for proving the consensus property of the system;
- A definition of a replicated state-transition machinery, implementing the per-node logic of the protocol, and semantics of the asynchronous network used for establishing protocol invariants;
- A formulated eventual consistency (global consensus) property for a blockchain network with a clique topology, a whole-system invariant implying the consensus in a quiescent state, and a proof of this invariant’s inductivity, i.e., preservation by the network semantics.

In this work, we focus exclusively on system safety properties, i.e., proving that “nothing goes wrong”. There are, indeed, more facts to establish about blockchain-based protocols, involving liveness (aka chain growth), probabilistic irrevocability, stronger notions of consistency, and various security properties [10, 19]. We do not address any of them in this paper, and consider statements and proofs of those properties as future applications of our formal model, discussing some of them in Section 7.

Our Coq development is publicly available online [27]:

https://github.com/certichain/toychain

1.2 Paper outline
In the remainder of the paper, we explain, by example, behaviours of blockchain-based networks and informally sketch intuition for system consistency (Section 2). We then describe the design and implementation of the core data structures, such as block forests, and their dependencies on the externally-provided security primitives in Section 3. In Section 4, we define the protocol machinery and the network semantics, elaborating on the statement and the proof of the consensus property in Section 5. We report on our mechanisation experience and lessons learned in Section 6. We then discuss limitations and possible future applications of our model in Section 7. We survey related verification and formalisation efforts in Section 8, and conclude in Section 9.

2 Overview
We begin by walking through an example that demonstrates interactions between nodes in a blockchain-based protocol and shows how consensus is achieved.

The goal of the consensus protocol is to guarantee that network participants agree wrt. the order in which transactions happened. This is achieved not by ordering transactions directly, but rather by grouping them into blocks and then agreeing, via FCR—a comparison operation on block sequences (chains), which resulting blockchain to adopt. Assuming an agreement upon the rules of the protocol and initial state of the system, (i.e., everyone has the same Genesis Block GB in the local state, as shown in Figure 1(a)), and provided everyone follows the rules, consensus is guaranteed once all block-carrying messages are delivered.
The stages (b)–(g) of Figure 1 demonstrate further interactions in a 3-node blockchain-based system. Each stage also provides, in the top-right corner, the implicit “globally shared” tree of blocks, which will eventually be replicated in each of the participants and will serve to compute the globally shared ledger, thus delivering the desired ledger consensus. At any point in time, participants may issue transactions, which they broadcast to all their peers. For instance, a node (1) creates and announces a transaction $tx_1$ to its peers in (b). The messages containing the transactions propagate throughout the network, and each node temporarily stores them in its local transaction pool ($\{tx_1\}$ in stage (c)). When “minting” (i.e., creating) a new block, nodes embed the transactions they know of inside the block and broadcast it to all participants in the network, as is done by node (3), which creates the block $A$ with the transaction $tx_1$ in (d), setting $A$’s “parent block” to be $GB$.

Since the system is distributed, multiple transactions can be issued and propagated concurrently. As a result, when a block is minted, the minter does not necessarily know of all pending transactions, as the messages containing them might still be in transit. This is what happens in stage (e), where the node (1), which creates block $B$, has yet no knowledge of $tx_2$. This delay in propagation also might cause certain transactions to be included in different, “conflicting” blocks. For example, in (f) the transaction $tx_3$ has been included in both block $B$, minted previously by node (1), and block $C$, which is newly-minted by node (2). As such, looking at the global tree view in the top-right corner of (f), $tx_3$ is part of two blockchains: $[GB, A, C]$ and $[GB, A, B]$. However, this is not an issue for the global agreement: as FCR imposes a total order on blockchains, it uniquely determines which of the candidate chains is the correct one—in this case, $[GB, A, C]$ (we decide so for the sake of this example, indicating this by the green colour in the “shared” tree). Thus, all transactions, including $tx_3$, appear only once within the correct chain. As more messages propagate, more participants agree on the “correct” $[GB, A, C]$ to be the “correct”, i.e., canonical blockchain (g), until finally, everyone is agreement when the system is in a quiescent state with no messages in transit.

In the illustrative example above, we have seen how the system evolves over time and how it resolves conflicts by using the globally known FCR function. It is crucial for the purpose of eventually reaching a consensus that the FCR imposes a total order on all possible blockchains, such that conflicts are uniquely settled. Also note that FCR is computed locally by all participants, with no communication needed. Intuitively, these two properties put together imply that if two participants have the same blocks in their local “block forests”, they will agree wrt. what the correct global chain is. Extended to an entire network, this means that when all blocks have been delivered, all nodes decide upon the same chain. Or, in other words, all nodes are in consensus.

In the following sections, we establish this formally.
Time, Addr \triangleq \mathbb{N} \\
\begin{align*}
  b \in \text{Block} &::= \{ \text{prev}: \text{Hash}; \text{txs}: \text{Tx}^*; \text{pf}: \text{Proof} \} \\
  c \in \text{Chain} &::= \text{Block}^* \\
  bf \in \text{BForest} &::= \text{Hash} \xrightarrow{\text{inj}} \text{Block} \\
  tp \in \text{TxPool} &::= \mathcal{P}(\text{Tx}) \\
\end{align*}

(a) Blocks, chains, and block forests

Hash, Proof, Tx : eqType
\begin{align*}
  GB &: \text{Block} \\
  hash_b &: \text{Block} \rightarrow \text{Hash} \\
  hash_t &: \text{Tx} \rightarrow \text{Hash} \\
  mkProof &: \text{Addr} \rightarrow \text{Chain} \rightarrow \text{option Proof} \\
  VAF &: \text{Proof} \rightarrow \text{Time} \rightarrow \text{Chain} \rightarrow \text{bool} \\
  FCR &: \text{Chain} \rightarrow \text{Chain} \rightarrow \text{bool} \\
  txValid &: \text{Tx} \rightarrow \text{Chain} \rightarrow \text{bool} \\
  txExtend &: \text{TxPool} \rightarrow \text{Tx} \rightarrow \text{TxPool} \\
\end{align*}

(b) Parameter primitive types and functions

Figure 2. Data structures and framework parameters.

3 Data Structures for Blockchains

We now present the data structures and primitives necessary for implementing the logic of the blockchain consensus protocol’s replicated state machines.

3.1 Parameters and core data structures

The top part of Figure 2 shows the definitions of the main data structures we are going to use. Some of the primitive data types are left undefined (cf. Figure 2 (b)), as they serve as parameters for the framework. For example, the types of time-stamps Time, necessary for modelling disciplines such as Proof-of-Stake, and network addresses Addr (ranged over by \( \tau \) and \( a \), correspondingly) are isomorphic to natural numbers \( \mathbb{N} \). At the same time, the types Hash of hash values, proof objects Proof and transactions Tx (ranged over by \( h \), pf, and tx, correspondingly) can be arbitrary (e.g., natural numbers, strings, etc) as long as they come with decidable equality checking, which is indicated by the eqType annotation.

Blocks are represented as records with three fields. The first one, prev, stores the hash of some block (even the very same one, although in most cases such blocks will be deemed ill-formed). The field txs stores a sequence of transactions, contained within this block, in the order they are supposed to take place.\(^2\) Finally, the proof object in pf is used to validate the block.

Two functions, hash\(_t\) and hash\(_b\), for computing hash-values of transactions and blocks correspondingly, are provided by the client of the framework. For the sake of brevity, in the remainder of the paper, we will use the overloaded notation \#\( x \) for computing the hash of a value \( x \), which is either a block or a transaction, using the corresponding hash-function. The only requirement imposed on the hash function is it being injective, as stated by the axiom hash\(_{inj}\) in Figure 3.

We require the client of the framework to provide a dedicated Genesis Block GB, which is going to serve as an initial “seed” for all local ledgers, and is globally known in the system. This block is slightly different from other blocks we will observe in the system, as it is a subject of the first two hypotheses (axioms) we impose, which are shown in Figure 3. First, the hash of GB should be equal to GB’s prev value (GB\(_{hash}\)). This requirement seems quite artificial, but it is easy to achieve in practice by redefining the result of a typical hash-function for just one value, and it simplifies reasoning about forests, essentially ensuring that there can be no parent block of GB, in the presence of possible cycles between other (ill-formed) blocks referring to each other. The second axiom GB\(_{txs}\) ensures that the genesis block contains an empty transaction sequence \([\ ]\).

Blockchains (or simply chains) are defined as sequences of blocks, and block forests (or forests) are encoded as partial finite maps from hashes to blocks. The notions of “valid” chains and forests will follow below.

The next two parameter functions mkProof and VAF work in tandem: the former is used to obtain a proof object for a specific node and on top of a particular chain, and it might fail, hence the option result type. In real-world blockchain implementations, computing a value of mkProof is an expensive operation, as it controls the number of valid blocks in the system, but here we do not model computational complexity, as it is irrelevant for establishing consensus, which is a safety, not a liveness property. Dually, VAF is used to validate proof objects for a chain and it also takes a system-provided time-stamp as an additional parameter. The only VAF-related axiom we need, VAF\(_{nocycle}\), ensures

\[
\begin{align*}
  \text{hash\(_{inj}\)} &: \forall x \ y, \ x \neq y \implies x = y \\
  \text{GB\(_{hash}\)} &: \text{prev} GB = \#GB \\
  \text{GB\(_{txs}\)} &: \text{txs} GB = [\ ] \\
  \text{VAF\(_{nocycle}\)} &: \forall b \ c \ \tau \ c, \ \forall c, \ c) \quad \in \text{Fig. 3).}
\end{align*}
\]

\[\begin{align*}
  \text{VAF\(_{nocycle}\)} &: \forall b \ c \ \tau \ c, \ \forall c, \ c) \\
  \text{FCR\(_{rel}\)} &: \forall c, \ c) \\
  \text{FCR\(_{trans}\)} &: \forall c, \ c) \\
\end{align*}\]

Figure 3. Axioms of the framework parameters.

\(^2\)For simplicity, we consider transactions located in the same block to be non-conflicting with each other.
that a freshly “minted” (i.e., created) block $b$, for which a proof has been obtained wrt. to an “underlying” chain $c$, cannot be contained in the same $c$. An opposite situation would be an anomaly, and does not occur in real situations, in part due to the practical rareness of hash collisions.

The client-provided function $FCR$ allows one to compare the weights of two chains. From now on, we will abbreviate $(FCR \ c_1 \ c_2)$ as $c_1 \succ c_2$. The axioms $FCR_{\text{rel}}$, $FCR_{\text{trans}}$, $FCR_{\text{nrefl}}$ ensure that the order $FCR$ imposes on chains is total, transitive, and irreflexive. Finally, the axiom $FCR_{\text{ext}}$ postulates that if a chain $c_1$ is a subchain of $c_2$ (i.e., $c_1 \cdot c_2 \triangleq c_2 = c' ++ c_1 ++ c''$ for some $c', c''$, such that at least one of $c'$ and $c''$ is non-empty), then $c_2$ is “heavier”.

Finally, the transaction validation function $txValid$ ensures the absence of conflicts between a transaction $tx$ and a preceding chain $c$, being always true for an empty chain (as asserted by $txValid\ nil$), and $txExtend$, which we did not have to constrain, is used to change a pool of pending transactions held by a particular node.

3.2 Largest chains and block forest evolution

Block forests are the main data structures nodes use to store incoming and locally minted blocks, and to reconstruct the actual ledger of transactions. The ledger of a forest $bf$, typeset as $[bf]$, is defined as the largest (wrt. $FCR$) chain starting at $GB$ and ending with some block $b$, which has a corresponding entry in $bf$.

How do we construct such a chain? To do so, we should restrict the class of forests $bf$ we are working with to those satisfying the following three properties:

1. $\forall h_1, h_2 \in \text{dom}(bf), h_1 = h_2 \Rightarrow bf(h_1) = bf(h_2)$;
2. $\forall h \ b. \ bf(h) = b \Rightarrow h = \#b$;
3. $bf(#GB) = GB$.

The first property states that every key in $bf$ uniquely identifies its entry; the second ensures that for every block-entry in $bf$, its key is a hash of the corresponding block; finally, the third property makes sure that $bf$ contains the Genesis Block with its key. We define a forest $bf$ satisfying 1–3 as valid($bf$) and will denote a block $b$ having a corresponding entry $\#b \mapsto b$ in a valid forest $bf$ as $b \in bf$, slightly abusing the $\in$-notation.

In the beginning of a system interaction, each node holds the same forest $bf_0 = \{#GB \mapsto GB\}$, which is trivially valid. As the nodes start minting new blocks and broadcast them, local forests might be extended with new blocks, for which we define the following operation:

$$bf \circ b \triangleq \begin{cases} bf & \text{if } \#b \in \text{dom}(bf) \text{ then } bf \cup \{\#b \mapsto b\} \\ \text{else } bf \cup \{\#b \mapsto b\} \end{cases}$$

That is, for any block, the result of $bf \circ b$ is valid, if so was $bf$. Thus, fixing $\circ$ as the only way to add a new block to a forest, in the rest of the paper we will be only dealing with valid forests, unless said otherwise.

Let us now compute the largest chain in a forest $bf$. Indeed, even a valid forest might not be a tree, due to gaps and possible cycles in the partial map that encodes it. While cycles do not seem to be plausible in a real-world setting, we model them anyway; the “gaps” are, on the other hand quite frequent, as they correspond to blocks received out-of-order. To be considered a ledger candidate, a chain $c$ should satisfy the following conditions:

1. It should contain no duplicate blocks;
2. For any block $b \in c$, $b = GB$ or $\text{prev } b = b'$, where $b'$ is the block preceding $b$ in $c$;
3. The first block of $c$ should be $GB$ ($gb\text{-founded } c$);
4. For any block $b \in c$, and any transaction $tx \intx b$, $txValid \ tx' c'$ should be True, where $c'$ is a prefix of $c$, preceding $b$ ($tx\text{-valid } c$).

To deliver such a candidate, we first construct a total function $\text{chain}(bf, b)$ that returns a chain $c$, ending with a block $b$ (or just one-element chain $[GB]$ if $\#b \notin \text{dom}(bf)$) satisfying the conditions 1 and 2 above by implementing a “backwards walk” by $\text{prev}$-links from $b$ in $bf$ and ensuring that we do not visit the same block twice. Such a walk terminates if we encounter a cycle in $bf$ or we reach $GB$, which is its own previous block. The exact code of $\text{chain}$ can be found in our supplementary Coq sources. Using $\text{chain}$, we construct candidates considering all blocks in $bf$ and choose the largest one from those that satisfy conditions 3 and 4. In set notation, for a valid $bf$, $[bf]$ is defined as follows:

$$[bf] \triangleq \max_{FCR} \{ c \mid c = \text{chain}(bf, b), b \in bf, \ gb\text{-founded } c \land \text{tx-valid } c \} \quad (3)$$

The function $[bf]$ is implemented to be total so it returns $[GB]$ as a default ledger, if no better one is found.

To get a better intuition on the dynamics of $[bf]$ as the forest $bf$ keeps being extended with new blocks, let us take a look at Figure 4, which shows several states of a valid block forest with $\text{prev}$-links depicted by gray arrows. The stage (a) depicts a valid forest whose ledger is $c = \{GB, A, B, C\}$, with all other chains being less heavy or prefixes of $c$. In stage (b), due to out-of-order arrival, a block $G$ has been added to the forest, but at that moment it is orphaned, hence a chain built from it is not $gb\text{-founded}$. Once the missing block $F$ arrives in stage (c), the forest gets a new ledger, namely $\{GB, D, E, F, G\}$, as it is more $FCR$-heavy than $\{GB, A, B, C\}$. Stages (d) and (e) show block forests that we account for in our implementation, but that will not correspond to local states of the protocol participants during a normal, non-Byzantine execution (we discuss Byzantine cases in Section 7). Specifically, the forest in (d) has a block $M$, which is non-$tx\text{-valid}$ wrt. $\{GB, A, B, C\}$, and therefore
does not contribute a ledger candidate, preventing all further chains in the forest that include it from being considered for the largest ledger. Finally, the case (e) demonstrates a cycle between two blocks, X and Y, so neither of them is included into any chain to be considered a ledger candidate. In Section 4, we will show how the protocol prevents cases (d) and (e) from happening, when all participants are faithful.

3.3 Local forests and system-wide forest union

The cases (a)–(c) from Figure 4 depict possible configurations of local forests, owned by particular participants of a blockchain protocol in the middle of a system’s execution, when some of the blocks have already been received, while some others are in “in-flight” messages, i.e., yet to be delivered. Furthermore, in the absence of Byzantine participants, a system-wide union $bf$ of all locally owned forests will be defined, valid, and furthermore satisfy the following property, dubbed $good(bf)$:

$$good(bf) \triangleq \forall b \in bf, \
\text{gb-founded}(\text{chain}(bf,b)) \land \text{tx-valid}(\text{chain}(bf,b))$$

(4)

In other words, a good forest is a tree, such that GB is its root, and a chain built from any of its blocks starts with GB and has no blocks with invalid transactions. A pleasant consequence of the fact $\text{gb-founded}(\text{chain}(bf,b))$ is that the chain has no gaps, i.e., the result of $\text{chain}(bf,b)$ will not be affected by adding new blocks to $bf$, which is stated formally by the following lemma:

Lemma 3.1. For a valid forest $bf$ and a block $b$, if $\text{gb-founded}(\text{chain}(bf,b))$ then for any block $b'$, $\text{chain}(bf \triangleleft b',b) = \text{chain}(bf,b)$.

The following theorem is key for showing that locally minting a new block, when done right, always increases the local ledger, and, thus, has a chance to increase the global one, i.e., the ledger of a system-wide forest union.

Theorem 3.2. If a forest $bf$ is valid and a block $b$ is such that $\text{tx-valid}([bf],b)$, $\text{prev} b = \#(\text{last} \{bf\})$, and $\text{VAF} (\text{pf} b) \tau ([bf])$ for some $\tau$, then $[bf \triangleleft b] > [bf]$.

That is, if a block $b$ is minted to extend the current $bf$’s ledger, adding $b$ will deliver a heavier one.

For a (multi-)set of blocks $bs = \{b_1,b_2,\ldots,b_n\}$ and a forest $bf$, we define the operator $bf \triangleleft bs$ as follows:

$$bf \triangleright= \{b_1,b_2,\ldots,b_n\} \triangleq \ldots((bf \triangleleft b_1) \triangleleft b_2)\ldots \triangleleft b_n \quad (5)$$

The definition (2) of $\triangleleft$ implies that reordering or duplication of blocks in $bs$ does not affect the result of $\cdot \triangleleft bs$.

We define the partial order $\sqsubseteq$ on valid forests as follows:

$$bf_1 \sqsubseteq bf_2 \triangleq \exists bs, bf_2 = bf_1 \triangleleft bs \quad (6)$$

We conclude this section with two theorems that are crucial for relating changes in a “node-local” ledger, due to minting a new block wrt. a node-local forest $bf$, and a “global” ledger built from the system-wide union of forests $\hat{bf}$, such that $bf \sqsubseteq \hat{bf}$. The first theorem states that, if a new block is global goodness-preserving, and if the new local ledger is heavier than the old global one, then the new local ledger is the new global one:

Theorem 3.3. For valid forests $bf$, $\hat{bf}$ and a block $b$, if $\text{good}(bf)$, and $\text{good}(\hat{bf} \triangleleft b)$, and $[bf \triangleleft b] > [\hat{bf}]$, and $bf \sqsubseteq \hat{bf}$, then $[bf \triangleleft b] = [\hat{bf} \triangleleft b]$.

The second theorem states that a correctly locally minted block $b$, if it does not contribute to create a ledger heavier than the current global one, will not change the global ledger even when added to the global forest:

Theorem 3.4. For valid forests $bf$, $\hat{bf}$ and a block $b$, if $\text{good}(\hat{bf})$, and $\text{good}(\hat{bf} \triangleleft b)$, and $\text{tx-valid}([bf],b)$, and $\text{prev} b = \#(\text{last} \{bf\})$, and $\text{VAF} (\text{pf} b) \tau ([bf])$, and $bf \sqsubseteq \hat{bf}$, and $[\hat{bf}] \geq [bf \triangleleft b]$, then $[bf \triangleleft b] = [\hat{bf}]$.

In a foresight of the development to be presented in the following sections, it should be intuitively clear that a ledger $[\hat{bf}]$ of a system-wide forest union is what consensus is going to be reached upon. That is, each node, if it follows the rules of minting and adopting blocks, will eventually “align” its local ledger with $[\hat{bf}]$. 

Figure 4. Different states of a valid block forest and its ledger chain (green) when extended with more blocks.
4 The Protocol and its Semantics

From the description of the core data structures, we proceed to outline the logic of the protocol itself, represented as a family of replicated state-transition systems which communicate by asynchronously exchanging packets.

4.1 System state-space

Figure 5 shows all state-space components of our system encoding. System configurations \( \sigma \) are pairs of a global state \( \Delta \) and a packet soup \( P \). The former is a finite partial mapping from node addresses to their local states; described further below, while the latter is a (multi-)set of packets.\(^3\) Packets are simply triples, with the first component being the message sender, the second is the address of the destination, and the third one is the message content. We will further refer to the destination address of a package \( p \) as \( dest \ p \).

A local state \( \delta \) of a node is a quadruple \((this, as, bf, tp)\). Its first component \( this \) is the address of the node itself, which coincides with the key of this node in the system’s global state; the second component is a set of addresses \( as \) representing the peers the node is aware of; the third component is a block forest \( bf \), described in detail in Section 3, used to store the minted and received blocks; finally, the last component is a pool of transactions \( tp \), storing locally created or received transactions to be included into minted blocks in the future.

Specific contents of the messages the nodes can send are defined by the data type \( \text{Msg} \) and include: \( \text{NullMsg} \), which has no effect and is a message-passing analogue which coincides with the key of this node in the system’s mechanisation they are modeled as multi-sets.


A data type \( \text{Instr} \) for instructions serves to encode these operations to encode node-and announcing a transaction, and minting a new block.

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4.2 Local node semantics

Figures 6 and 7 show the per-node transitions. While in our implementation they are encoded as executable Coq functions, in this paper we follow a more traditional relational style of presenting an operational semantics.

We split the semantics into receive-transitions and internal transitions. The former ones are of the form \( \delta \xrightarrow{p,(\delta',ps)} \) taking a node from a state \( \delta \) and to \( \delta' \) when processing a package \( p \), also emitting a new set of packages \( ps \). The latter ones, \( \delta \xrightarrow{(i,\tau)} (\delta',ps) \), describe change of a state from \( \delta \) and \( \delta' \) with emission of packages \( ps \) as a result of executing the instruction \( i \) at a globally synchronised time \( \tau \).

The receive-transitions (Figure 6) follow the intuition of the corresponding messages, and are mostly straightforward, so we only describe a few in prose. When taking a RcvAddr step, a node not only adds the new addresses to its local pool, but also sends a ConnectMsg request to the new peers it has learned about \((ps_1)\) and propagates the new information to its current peers \((ps_2)\), which will then themselves connect to the new peers (if they hadn’t already). When receiving a new

\(^3\)Our semantics is resilient with respect to packet duplication, so here for simplicity we assume packet soups to be sets, while in or mechanisation they are modeled as multi-sets.
transaction or a block (via RcvTx or RcvBlock), a node informs its peers of now possessing it by sending an InvMsg, potentially causing a lot of redundant messages, which are nevertheless handled without any concerns for safety. Finally, the last two transitions serve to inform a node of new transactions and blocks in the system (via RcvInV), so it could request them by sending a GetDataMsg message, and the response to it will be sent (via RcvGetData) in the form of TxMsg or BlockMsg.

It is perhaps slightly non-obvious, but the rules allow to model the possibility of a node “joining late” and eventually “catching up” with the rest of the system, thanks to RcvConnect and other transitions that send known information about blocks transactions to the package origin from, so it could request them via RcvInv.

Figure 7 shows the two internal transitions that are triggered by the corresponding instructions. The IntTx simply adds a new transaction to the local pool, so it could be included into a block later, and announces it to the node’s peers. The IntMint transition relies on the block forest machinery and related primitives described in the previous section. Specifically, we (rather optimistically) assume that a node locally checks the new minted block b with respect to its prefix chain, before adding it to its local forest and sending it to its peers.

With the rules in Figures 6 and 7, we intentionally define a non-optimal version of the protocol, such that nodes executing the transitions populate the packet soup with a lot of redundant messages. Yet, as we will show in Section 5, this does not pose problems for establishing consensus on the state of the global ledger.

### 4.3 Network semantics

The network semantics rules, parameterised by a selector s, are shown in Figure 8. They are standard for modeling interleaved concurrency with non-deterministic internal choices and message delivery. The three rules account for a possibility of delivering a randomly picked package p from the soup P to a destination a (NetDeliver), a node a taking an internal step with an instruction i (NetInternal) or doing nothing (NetIdle).

While the rules do not change the global set of node addresses, we nevertheless can model a scenario of a node “joining” the network, assuming that it already has a predefined address and a correctly initialised initial

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**Figure 6.** Local semantics, Part I: receive-transitions.

**Figure 7.** Local semantics, Part II: internal transitions.
Mechanising Blockchain Consensus

<table>
<thead>
<tr>
<th>Network transitions: $⟨Δ, P⟩ \xrightarrow{\text{NetDeliver}} ⟨Δ', P'⟩$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p \in P$, dest $p = a$</td>
</tr>
<tr>
<td>$Δ(a) = δ$</td>
</tr>
<tr>
<td>$δ \xrightarrow{p} (δ', ps)$</td>
</tr>
<tr>
<td>$\langle Δ, P \rangle$</td>
</tr>
<tr>
<td>$\xrightarrow{\text{Send} a}$</td>
</tr>
<tr>
<td>$\langle Δ[a \rightarrow δ'], P \setminus {p} \cup ps \rangle$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>NetInternal</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Δ(a) = δ$</td>
</tr>
<tr>
<td>$δ \xrightarrow{(i, τ)} (δ', ps)$</td>
</tr>
<tr>
<td>$\langle Δ, P \rangle$</td>
</tr>
<tr>
<td>$\xrightarrow{\text{SendInt} a \in V \setminus {i}}$</td>
</tr>
<tr>
<td>$\langle Δ[a \rightarrow δ'], P \cup ps \rangle$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>NetIdle</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\langle Δ, P \rangle$</td>
</tr>
<tr>
<td>$\xrightarrow{\text{Idle}}$</td>
</tr>
<tr>
<td>$\langle Δ, P \rangle$</td>
</tr>
</tbody>
</table>

The desired theorem is as follows:

$$\text{Coh}(⟨Δ, −⟩) \triangleq \forall a \in \text{dom}(Δ), \exists as \ b f \ tp, \Delta(a) = \langle a, as, bf, tp \rangle \land \text{valid}(bf) \quad (8)$$

The validity of each local forest $bf$ is via the definition (1). Any of the $σ_0$ we consider satisfies it, and the property Coh is inductive, because all manipulations with node-local block forests are done using the $<$ operation (2).

### 5.2 Eventual ledger consistency

Let us now formulate the eventual consistency of the system. Informally, it says that when there are no in-flight messages between any of the nodes, they all should agree on the local ledger, which can be, thus, thought of as a globally shared one [31].

In practice, however, communication between nodes never stops. Our protocol features many “modes of communication” (announcing a block, requesting hashes, etc), and, as it turns out, not all of them should be ceased for reaching consensus on ledgers. What is essential is to have no in-flight instances of BlockMsg.\(^5\) Having no in-flight block-messages, however, is not the only requirement for the universality of the consensus (i.e., ensuring that each two nodes have the same ledger): it might be the case that some nodes joined late, and due to the delays in updating the topology, have not yet requested all missing data from their peers. Characterising consistency conditions in this case would require us to take the “late joiners” into account. While not impossible, this would make the whole consistency statement quite complicated. To avoid this, in this paper we decided to formulate the consistency in a simpler setting: a clique network topology, restricting the initial configurations to those where every node’s known peers include all addresses in the global state.\(^6\)

We embed the clique topology assumption into the whole-system property Cliq, whose formal definition we postpone until Section 5.3. For now, let us present the eventual consistency result it implies. For this, we introduce two auxiliary definitions. The first one extracts a ledger for a node $a \in \text{dom}(Δ)$ from a global state $Δ$.

$$\text{ledger}(Δ, a) \triangleq [bf], \text{ s.t. } \langle a, −, bf, − \rangle = Δ(a) \quad (9)$$

The second returns all in-flight blocks for $a$ in a soup $P$:

$$\text{blocksFor}(P, a) \triangleq \{ b | \langle −, a, \text{BlockMsg} b \rangle \in P \} \quad (10)$$

The desired theorem is as follows:

\(^4\)We could have added another internal transition rule for emitting a ConnectMsg, but this is orthogonal to our study of system safety.

\(^5\)The version we present is a form of quiescent consistency [1, 4].

\(^6\)This situation is quite common for corporate blockchain-based protocols, where all peers know each other from the very beginning.
Theorem 5.1 (Consensus in a clique topology). For \( \sigma = (\Delta, P) \), if Cliq(\( \sigma \)) holds, then there exists a chain \( c \), such that for any node \( a \in dom(\sigma) \), \( c \geq ledger(\Delta, a) \) and \( blocksFor(P, a) = \emptyset \) implies \( ledger(\Delta, a) = c \).

The chain \( c \) from Theorem 5.1 statement is a globally shared ledger, and in a quiescent state, all nodes have it.

5.3 Clique invariant

We now show the statement of the Cliq property, highlight its key insights, and convey the intuition of the proof that it is indeed inductive for systems that start in initial configurations \( \sigma_0 \) with a clique network topology.

The formal definition of Cliq, with the most important conjuncts labelled (i)–(iv) is given in Figure 9. The first two non-labelled conjuncts ensure that the property holds over configurations that are coherent (8) and have a clique topology, as discussed above. The rest of the definition is more interesting, as it exhibits an important property of blockchain-based protocols, which we call the law of block conservation. The “conservation” is expressed via the existence of a global forest \( b^\mathbf{f} \) (foreshadowed as a system-wide forest union in Section 3.3), which is a superset of the local forest of any node \( a \), as stated by conjunct (i), and can be obtained by adding all blocks currently in-flight towards \( a \) to \( a \)'s local forest. The global forest \( b^\mathbf{f} \) has the “canonical” ledger \( c \) (ii), which is larger or equal than any local ledger (iii). Finally, there is always a node \( \bar{a} \in dom(\Delta) \) that has the canonical ledger \( c \), even though \( \bar{a} \)'s local forest might be a strict subset of \( b^\mathbf{f} \) (iv).

The statement of Theorem 5.1 trivially follows from (i)–(iii) as then the subject node’s forest is exactly \( b^\mathbf{f} \).

Why is Cliq inductive? In our Coq development, we have proved that Cliq is preserved by the network semantics. The proof is of interest, as it heavily relies on the idempotence of the \( \mathord{\prec} \) operation, and the “goodness” of the global forest \( b^\mathbf{f} \), whose ledger \( c \) (owned by at least one node in the network) serves as the constructive witness of what the consensus is going to be reached upon. The trickiest parts of the proof concern “restoring” conjuncts (ii) and (iv) when an arbitrary node takes the IntMint transition, with a chance of either (a) becoming the new owner \( \bar{a} \) of the global ledger \( c \), or (b) minting a block that is already in \( b^\mathbf{f} \) or simply does not deliver a heavier chain. The case (a) is handled by Theorem 3.3, while the case (b) is what is delivered by Theorem 3.4. The following theorem therefore holds:

Theorem 5.2. For systems that initially have a clique network topology, Cliq is an inductive invariant.

On the clique assumption. What would the invariant and the eventual consistency statements look like without the clique assumption? At the moment, the definition of Cliq ensures that for any node \( a \) and any block \( b \) in the system, \( b \) is either already in \( a \)'s local forest or is “flying towards” it. Now let us consider a case of a node \( a' \) that has joined late, announcing itself (via ConnectMsg) only to a few other participants. Then, \( a' \) might not have yet requested or has not yet been “forwarded” all the blocks already minted in the system. Therefore, in order to relate its local state to \( b^\mathbf{f} \), we would have to enhance the invariant with a conjunct expressing ongoing “propagation” of the known peers in the system, and replace (i) by it. In addition to that, we would need to consider situations when the topology is not a connected graph, in which case several “canonical” chains would co-exist without ever being reconciled. Stating the consensus property in such settings is our future work.

6 Elements of our Mechanisation

We mechanised all results described in this paper in Coq, making use of the Ssreflect/MathComp libraries [21, 29]. Our implementation of block forests builds on the library of partial finite maps by Nanevski et al. [23]. The size of our contributed codebase is pleasantly small, as demonstrated by the lines of code figures in Table 1.

In the proofs, we heavily relied on the small-scale reflection and rewriting machinery provided by Ssreflect [11]. For instance, our implementation of block forests features both constructive and computable definitions of prefix/fork relations on chains, as well as the corresponding reflect-view lemmas, to switch between the two representations. The definitions of all operations and predicates on block forests, such as (1)–(3), and on network configurations, such as (9) and (10), are also made decidable/computable. This design choice has paid off not only in reducing proof sizes, but also in the robustness of our proof scripts in the face of changes made in the definitions, which is surprising given how modest the amount of automation we used in the project was.
We now discuss the limitations of our protocol model. That is, participants in the network advertise their cious actors. Our invariant is not resistant to Byzantine hardware malfunctions or through the actions of mali-
trary behaviour. These may arise due to software bugs, Byzantine faults, in which participants exhibit arbi-
trary updates its state and advertises. none of them is, the process repeats the next time a peer is sufficient that one of these messages gets delivered. If advertised it. For reaching the consensus eventually, it information they do not have from all knowledge every time they update it, and they request additional communication redundancy. This scenario, which is very similar to that of dropped or of participant faults. In practice, the clique assumption means that we can largely ignore crash faults, and we do not expect to receive any responses.

A possible complication arises when a participant crashes while in the process of broadcasting a newly-
minted block, such that some peers receive it and others do not. This scenario, which is very similar to that of dropped packets, is difficult to accommodate in the current invariant. However, the problem with dropped packets would essentially disappear once we start making use of the protocol’s peer-to-peer facilities, as they a provide a large amount of communication redundancy. That is, participants in the network advertise their entire knowledge every time they update it, and they request information they do not have from all peers that have advertised it. For reaching the consensus eventually, it is sufficient that one of these messages gets delivered. If none of them is, the process repeats the next time a peer updates its state and advertises.

**Byzantine behaviours.** A special case of faults is that of Byzantine faults, in which participants exhibit arbitrary behaviour. These may arise due to software bugs, hardware malfunctions or through the actions of malicious actors. Our invariant is not resistant to Byzantine faults. For example, the proof relies on the fact that all blocks in the canonical block forest are \( \text{tx-valid} \). This is true under normal operation, but can be invalidated at will by a malicious actor. In order to reason about the ineffectiveness of Byzantine faults, we will have to introduce to the invariant some notion of honest participants being in the majority and in communication with each other, and to find a way of accommodating within the proofs the presence of “bad” blocks.

It is important to stress that these invalid blocks, as seen in cases (d) and (e) of Figure 4, do not in any way prevent the protocol from operating correctly (from the perspective of faithful participants), but merely make it more difficult to prove that it does.

**Other protocol properties.** In this work, we have focused exclusively on the safety of the system, i.e., the property that all correct nodes agree wrt. which ledger they adopt. Other properties, such as liveness and various security properties, depend on the choice of system parameters \( \text{hash} \), \( \text{VAF} \), and \( \text{FCR} \). For example, system security almost certainly requires that we use a cryptographic hash function, i.e., a hash function that is both collision-resistant (approx. injective) and pre-image resistant (given \( h(m) \), finding \( m \) is computationally hard). Moreover, we likely want \( \text{VAF} \) to impose a reasonable delay between consecutive block mintings, such that messages have time to propagate throughout the network—this would provide an adequate quiescent state, and thus consensus can be reached. Studying the full implications of different framework parameters is left for future work.

**Towards a verified blockchain implementation.**

Our implementation of the protocol is intentionally non-optimal. Whenever faced with a decision of how to implement a function, we always chose simplicity over efficiency. That being said, all of our functions of, e.g., processing block forests, are pure, so that they can be replaced with more efficient functional or imperative equivalents. For example, the reference implementation of \( \text{chain} \) performs a lot of redundant computation and would greatly benefit from a memoization strategy. Similarly, the message propagation strategy is very inefficient, and could possibly be replaced with a more sensible one.

Because our mechanisation of the protocol is encoded as a library of computable Coq functions that implement state transformers and message handlers, it should be possible to extract it to OCaml and run on top of a trusted shim implementation, thus providing a formally verified blockchain implementation, in the same way it has been done in the recent work on the DISEL framework [30, 33]. This setup, however, appears quite naïve and would be problematic in a realistic case of Byzantine faults, as the safety results established in this work hold only as long as all participants follow the protocol and

### Table 1. Sizes of definitions and proofs (LOC).

<table>
<thead>
<tr>
<th></th>
<th>Definitions</th>
<th>Proofs</th>
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<tbody>
<tr>
<td>Block Forests</td>
<td>579</td>
<td>1406</td>
</tr>
<tr>
<td>Protocol and Network</td>
<td>409</td>
<td>263</td>
</tr>
<tr>
<td>Consensus Properties</td>
<td>241</td>
<td>273</td>
</tr>
</tbody>
</table>

As an anecdote from our experience, while preparing this submission, a few days before the CPP’18 deadline and already after having completed proofs of all invariants, we have noticed an odd encoding of the rule \( \text{RcvConnect} \). In our mechanism to date, the corresponding transition was only adding the sender from to the local list of peers, but, rather selfishly, did not send a list of available hashes back as an \( \text{InvMsg} \). We have changed the implementation so it would precisely match the rule from Figure 6, and, to our surprise, *proofs of invariants broke*. We consider this an encouraging sign to invest more time into domain-specific automation for proofs about replicated state-transition systems.

## 7 Discussion

We now discuss the limitations of our protocol model and the implications of the assumptions we made.

**Network semantics and system faults.** As defined in Section 4, our network semantics is quite restricted. For instance, it does not include notions of packets being dropped or of participant faults. In practice, the clique assumption means that we can largely ignore crash faults, as we do not need other participants to relay our messages and we do not expect to receive any responses.

A possible complication arises when a participant crashes while in the process of broadcasting a newly-
minted block, such that some peers receive it and others do not. This scenario, which is very similar to that of dropped packets, is difficult to accommodate in the current invariant. However, the problem with dropped packets would essentially disappear once we start making use of the protocol’s peer-to-peer facilities, as they a provide a large amount of communication redundancy. That is, participants in the network advertise their entire knowledge every time they update it, and they request information they do not have from all peers that have advertised it. For reaching the consensus eventually, it is sufficient that one of these messages gets delivered. If none of them is, the process repeats the next time a peer updates its state and advertises.
use exactly the same version of the shim, which is hard to guarantee in an adversarial distributed environment.

8 Related Work

The results we presented in this paper are related to the latest advances in the areas of computer security, formal methods, and distributed systems.

Consistency of blockchain protocols. In the past few years, there has been a lot of interest within the security and privacy community for studying notions of consistency in application to blockchain protocols.

Garay et al. considered the core protocol underlying Bitcoin [22], focusing on its two properties, dubbed Common Prefix and Chain Quality [10]. The common prefix property is a probabilistic version of the notion of eventual consistency we have established in Section 5 of this paper. Specifically, they establish that all honest parties in the system agree on a common ledger prefix up to $k$ last blocks, where $k$ is a parameter of the system. The chain quality property tackles a Byzantine setting in which malicious participants may contribute ill-formed blocks, and states that the number of such blocks in the system is not very large, given that the majority of participants remain honest and follow the protocol. Unlike our formalization, the work by Garay et al. takes into account possible adversarial behaviours of the protocol participants, but restricts the communication to fully synchronous, i.e., messages in the system are instantly delivered without delays, whereas we allow for arbitrary delays and permutations in message delivery.

While that work focuses on proving the properties of Nakamoto’s consensus based on Proof-of-Work [22], in a follow-up to that result, Kiayias et al. propose a blockchain consensus protocol based on Proof-of-Stake [2] and possessing the same properties, and also a new one, Chain Growth, which ensures overall liveness for the honest parties [19], under the assumption of synchronous message delivery in the network. Finally, in a recent work, Pass et al. provided probabilistic boundaries with respect to chain growth and quality, as well as the analysis of other consistency and liveness properties of blockchain consensus in a fully asynchronous environment [25].

In contrast with those and many other works [6, 10, 19, 25] that analyse blockchain consensus from the perspective of security properties, thus, focusing on probabilistic reasoning about a protocol modeled as a composition of distributions, we present a simple operational model that immediately provides an executable semantics of the system, but only allows us to prove “coarse-grained” correctness conditions, such as eventual consistency.

None of the proofs of security properties of blockchain consensus we are aware of were mechanised.

Formal methods for blockchain applications. To date, the interest of the formal methods community wrt. blockchain-based systems is predominantly in applications of the technology, rather than reasoning about properties of the underlying protocols.

In the past two years, a number of works have been published on formal modeling and verification of smart contracts—a mechanism to associate executable code with certain blockchain transactions, providing a machinery for trusted decentralised arbitration, which gained a lot of attention thanks to its highly influential implementation in Ethereum [35]. Various formal semantics of Ethereum Virtual Machine (EVM) and its contract language Solidity were implemented in Coq [17], Isabelle [16], F* [3], Idris [26], K [14], Why3 [9], and in custom tools for static and dynamic analysis [20].

An implementation of an efficient data structure for transaction ledgers has been developed and verified in Coq by White [32], yet it has not been used in the context of verifying a protocol that employs this structure.

At the level of reasoning about protocols, Hirai has formalised a simple variant of a Proof-of-Stake protocol in Isabelle [15], proving a version of the protocols’ accountable safety: if two conflicting blocks get adopted in a shared block tree, then at least $\frac{1}{3}$ of participants may lose their entire deposits (stakes). This property is specific to Ethereum’s Casper protocol [5] and is orthogonal to the consensus result we established in this work.

Verification of distributed consensus. Several recent major efforts have fully mechanised and verified implementations of more traditional consensus protocols, such as versions of Paxos [7, 13, 18, 24, 28], Raft [34, 36], or the classical Two-Phase Commit [30, 33]. Even though none of those works consider blockchain consensus, we believe, many of those frameworks can handle it, as long as they can adopt our model and support reasoning about block forests. Therefore, we see our main conceptual contribution in distilling the protocol semantics and outlining the proof layout for blockchain consensus.

9 Conclusion and Future Work

In this work, we have presented a formal operational model of a distributed blockchain-based consensus protocol, implemented its core data structures, characterised the primitives it relies upon, and mechanically proved a form of the protocol’s eventual consistency, i.e., that a system that implements it does indeed reach a consensus.

In the future, we are going to enhance our mechanisation for reasoning about relevant security properties [19], modeling schedule-providing oracles as probabilistic distributions. We also plan to define an operational semantics for transactions run on top of the protocol, using it as a foundational platform for verified smart contracts.
References


