Programs and Proofs

Mechanizing Mathematics with Dependent Types

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Foreword

Formal Mathematics
as a branch
of Computer Science

Computer Science

Computer Science

- Computation and complexity
- Information and coding theory
- Data structures and algorithms
- Programming languages
- Formal methods and logics
- Security and cryptography
- Computer networks
- Databases
- Artificial Intelligence
- Computer graphics
- Computer/Human interaction
- Computer architecture
- Software Engineering
- ...

Mathematics

Mathematics

— *is what mathematicians do.*

Formal Mathematics

Formal Mathematics

makes rigorous statements…

Formal Mathematics

makes rigorous statements…

… and proves them.

What is a *proof* ?

Poor definition

A proof is sufficient evidence
or an argument for the truth of a proposition.

Poor definition

A proof is sufficient evidence
or an argument for the truth of a proposition.

Better definition

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each of which is either validly
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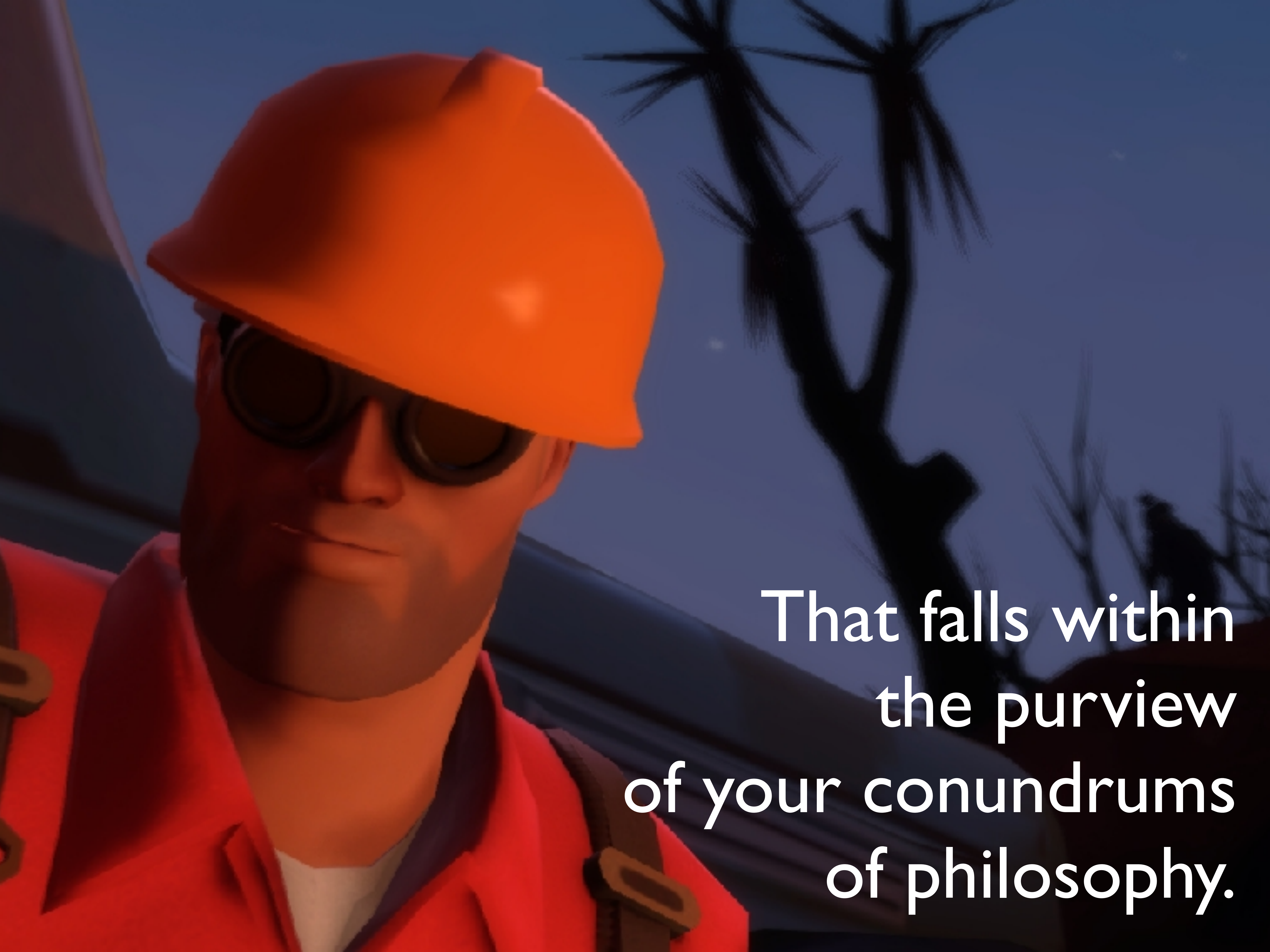
A proof is a sequence of statements,
each of which is either validly
derived from those preceding it
or is an axiom or assumption,
and the *conclusion* of which, is the statement
of which the *truth* is thereby established.

What is the *truth*?


That falls within
the purview
of your conundrums
of philosophy.

A proposition is considered to be *true*
(in a given set of assumptions and axioms)
if a proof of it can be *constructed*.

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if a proof of it can be *constructed*.

(so the truth *constructive* is *relative*)

What about *falsehood*?

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A proposition is *false*
if no proof can be *derived* for it.

How do we *construct* proofs?

How do we *construct* proofs?

By using *rules* and *hypotheses*.

A hypothesis

Assuming that the proposition *A* is true, we can derive that *A* is true.

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In formal logical notation: $\boxed{A \vdash A}$

Implication introduction

If, assuming that the proposition A is true, we can derive the proof of the proposition B, then we can derive the implication $A \Rightarrow B$.

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Modus ponens

The proposition *A* is true, and, moreover, *A* being true implies that *B* is true; then we can derive that *B* is true.

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$$\boxed{\dfrac{\vdash A \qquad \vdash A \Rightarrow B}{\vdash B}}$$

Conjunction introduction/elimination

From "A is true" and "B is true", we can derive that $A \wedge B$ is true as well.

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From "$A \wedge B$ is true" we can derive that *A* is true and that *B* is true.

$$\frac{\vdash A \wedge B}{\vdash A} \qquad \frac{\vdash A \wedge B}{\vdash B}$$

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$$\boxed{\frac{\vdash A}{\vdash A \vee B}} \qquad \boxed{\frac{\vdash B}{\vdash A \vee B}}$$

From "$A \vee B$ is true", "$A \Rightarrow C$" and "$B \Rightarrow C$", we can derive that C is true (*case analysis*).

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$$\boxed{\dfrac{\vdash A \vee B \quad \vdash A \Rightarrow C \quad \vdash B \Rightarrow C}{\vdash C}}$$

Universal quantification

If c is an *arbitrary* element of X, and $A(c)$ is true, then $\forall x \in X, A(x)$ is true (*universal generalization*).

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If c is an arbitrary element of X and $\forall x \in X, A(x)$ is true, then $A(c)$ is true (*instantiation*).

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$$\frac{\vdash \forall x \in X, A(x) \qquad c \in X}{\vdash A(c)}$$

Existential quantification

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If $\exists x \in X, A(x)$ is true, and for any $c \in X$, $A(c)$ implies B, then B is true (*generalized case analysis*).

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If $\exists x{\in}X, A(x)$ is true, and for any $c{\in}X$, $A(c)$ implies B, then B is true (*generalized case analysis*).

$$\frac{\vdash \forall x{\in}X, (A(x) \Rightarrow B) \qquad \vdash \exists x{\in}X, A(x)}{\vdash B}$$

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No introduction rule.

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Assuming falsehood, we can derive a proof of any statement.

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Special proposition: **False.**

No introduction rule.

Assuming falsehood, we can derive a proof of any statement.

$$\boxed{\frac{\vdash \textbf{False}}{\vdash A}}$$

Falsehood

A is *false* ($\neg A$) if, assuming *A* is true, we can derive the proof of **False**.

$$\boxed{\neg A \stackrel{\text{def}}{=} A \Rightarrow \textbf{False}}$$

Falsehood

Any statement can be proved to be true, assuming a *contradiction* ($A \wedge \neg A$).

$$\frac{\vdash A \wedge \neg A}{\vdash B}$$

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Any statement can be proved to be true, assuming a *contradiction* ($A \land \neg A$).

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Example 1: Naïve set theory is inconsistent (*Russel's paradox*).

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That, is it does not contain *paradoxes*.

Example 1: Naïve set theory is inconsistent (*Russel's paradox*).

Example 2: Zermelo–Fraenkel set theory is consistent (see *axiom schema of specification*).

How do we *check* proofs?

How do we *check* proofs?

By verifying *each* inference step.

This might be difficult

This might be difficult

(but not as difficult as to construct a proof)

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(but not as difficult as to construct a proof)

… but still

Some proofs are too critical

- Hardware correctness
- Software correctness
- Integrity of cryptographic protocols

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  - The proof is about 150 pages of handwritten math
- *Odd order theorem* (stated in 1911, proved in 1962)
  - The proof is about 250 pages of printed text
- *Four colour theorem* (proved in 1976)
  - 1936 special cases are discharged via *a program*

Can we use computers to check our proofs?

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Yes

Can we use computers to check our proofs?

Yes

In fact, this is what
some programmers do every day.

Programs

Programs

=

Data Types + Functions

Programming Languages

- Haskell
- ML
- F#
- Lisp
- Scala
- Agda
- Coq
- Perl
- Python
- Ruby
- C++
- Java
- C#
- …

Functional Programming Languages

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Functional Programming Languages

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- Data types define *immutable* values
- Functions are *values* as well
- Programs are *pure* functions

Statically-typed Functional Programming Languages

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Statically-typed Functional Programming Languages

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- Every value has a *type*
- *Type* defines a *set* of values
- Type of a program — its specification

A type with one element

```
Datatype unit := tt.
```

A type with one element

```
Datatype unit := tt.
```

$$unit \stackrel{\text{def}}{=} \{\ tt\ \}$$

A type with two elements

```
Datatype bool := true | false.
```

A type with two elements

```
Datatype bool := true | false.
```

$$\mathit{bool} \stackrel{\text{def}}{=} \{\, \mathit{true}, \mathit{false} \,\}$$

A type with no elements

```
Datatype empty := .
```

A type with no elements

```
Datatype empty := .
```

$$\mathit{empty} \stackrel{\text{def}}{=} \{\ \}$$

A type defined recursively

```
Datatype nat := 0 | .+1 of nat.
```

A type defined recursively

$\boxed{\texttt{n} \in \texttt{nat} \Rightarrow \texttt{n.+1} \in \texttt{nat}}$

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$\boxed{\texttt{n} \in \texttt{nat} \Rightarrow \texttt{n.+1} \in \texttt{nat}}$

$$\texttt{\textbf{Datatype} nat := 0 |} \overbrace{\texttt{.+1 \textbf{of} nat.}}$$

$$nat \stackrel{\text{def}}{=} \{\, 0, (0.+1), (0.+1.+1), \ldots \,\}$$

A type defined recursively

$$\boxed{\texttt{n} \in \texttt{nat} \Rightarrow \texttt{n.+1} \in \texttt{nat}}$$

```
Datatype nat := 0 | .+1 of nat.
```

$$nat \stackrel{\text{def}}{=} \{\, 0, \underbrace{(0.{+}1)}_{\mathbf{1}}, \underbrace{(0.{+}1.{+}1)}_{\mathbf{2}}, \ldots \,\}$$

A parametrized type

```
Datatype prod A B := pair of A & B
```

A parametrized type

```
Datatype prod A B := pair of A & B
```

$$A \times B \stackrel{\text{def}}{=} \{\ (a, b) \mid a \in A, b \in B\ \}$$

Another parametrized type

```
Datatype sum A B := inl of A | inr of B
```

Another parametrized type

```
Datatype sum A B := inl of A | inr of B
```

$$A + B \stackrel{\text{def}}{=} \{\ (a, 1)\ |\ a \in A\ \} \cup \{\ (b, 2)\ |\ b \in B\ \}$$

Parametrized recursive type

```
Datatype list A := Nil | Cons of A & (list A).
```

Parametrized recursive type

```
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```

$$\mathit{list}\ A \stackrel{\text{def}}{=} \mathit{Nil} \cup \{\ \mathit{Cons}(\mathrm{a}, \mathrm{l}) \mid \mathrm{l} \in \mathit{list}\ A\ \}$$

A simple function

```
Function negate : bool -> bool :=
  fun b => match b with
    | true  => false
    | false => true
  end.
```

A simple function

negate ∈ bool → bool

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Function negate : bool -> bool :=
  fun b => match b with
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  end.
```

A recursive function

```
Function even : nat -> bool :=
  fun n => match n with
    | n'.+1 => negate (even n')
    | 0     => true
  end.
```

A recursive function

$\boxed{\texttt{even} \in \texttt{nat} \to \texttt{bool}}$

```
Function even : nat -> bool :=
  fun n => match n with
    | n'.+1 => negate (even n')
    | 0     => true
  end.
```

An ill-typed function

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  end.
```

```
Wrong type: bool expected, but list ? found.
```

Types are program specifications

A type-checking algorithm ensures that each *value* has an appropriate *type*, i.e., that it belongs to the *corresponding set.*

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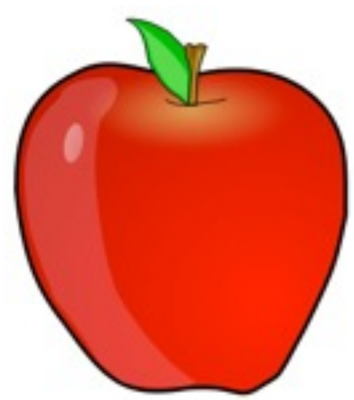

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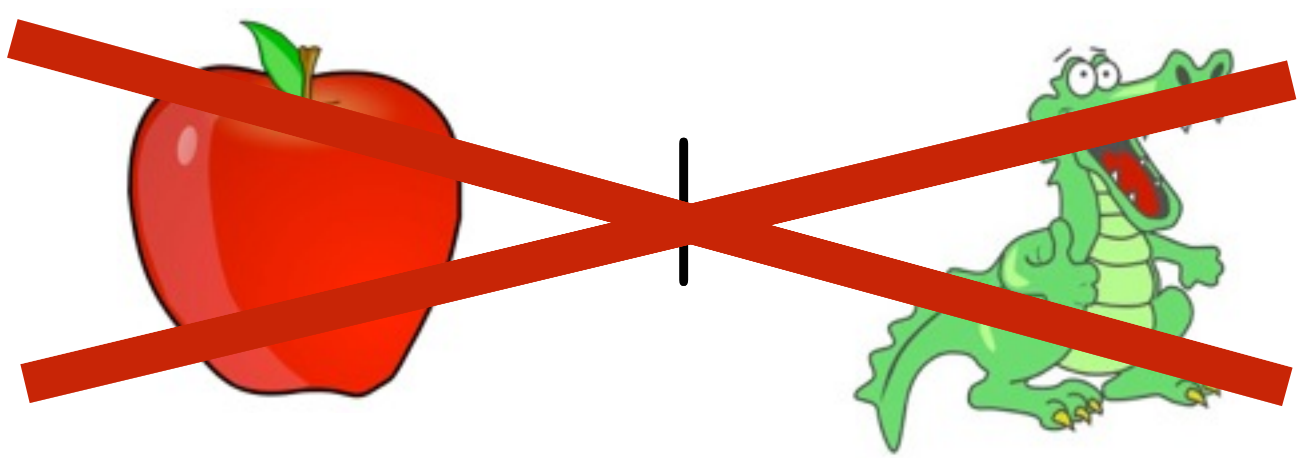

Given a type **A**, can we construct a program (value) **p**, such that **p** is an element of type **A**?

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In other words: can we *inhabit* the type **A**?

unit

```
unit
```

```
tt
```

bool

```
bool
```

```
true
```

bool

false

nat

`nat`

0

nat

1

nat

2014

empty

empty

???

```
A -> A
```

```
A -> A
```

```
fun (a: A) => a
```

```
A -> A
```

```
fun (a: A) => a
```

$$\frac{A \vdash A}{\vdash A \Rightarrow A}$$

```
A -> (A -> B) -> B
```

```
A -> (A -> B) -> B
```

```
fun (a: A)
    (f : A -> B) => f a
```

```
A -> (A -> B) -> B
```

```
fun (a: A)
    (f : A -> B) => f a
```

$$\frac{\vdash A \qquad \vdash A \Rightarrow B}{\vdash B}$$

```
(A × B) -> A
```

$$(A \times B) \to A$$

```
fun (a: A × B) =>
  match a with
  | pair a b => a
  end
```

```
(A × B) -> A
```

```
fun (a: A × B) =>
  match a with
  | pair a b => a
  end
```

$$\frac{\vdash A \land B}{\vdash A}$$

```
(A × B) -> B
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fun (a: A × B) =>
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(A × B) -> B
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```
fun (a: A × B) =>
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  | pair a b => b
  end
```

$$\frac{\vdash A \wedge B}{\vdash B}$$

```
A -> B -> (A × B)
```

```
A -> B -> (A × B)
```

```
fun (a: A)(b: B) => pair a b
```

```
A -> B -> (A × B)
```

```
fun (a: A)(b: B) => pair a b
```

$$\frac{\vdash A \qquad \vdash B}{\vdash A \wedge B}$$

```
A -> (A × B)
```

$$A \to (A \times B)$$

???

```
A -> A + B
```

```
A -> A + B
```

```
fun (a: A) => inl a
```

```
A -> A + B
```

```
fun (a: A) => inl a
```

$$\frac{\vdash A}{\vdash A \vee B}$$

```
(A + B) -> (A -> C) -> (B -> C) -> C
```

```
(A + B) -> (A -> C) -> (B -> C) -> C
```

```
fun (x: A + B)(f: A -> B)(g: B -> C) =>
  match x with
  | inl a => f a
  | inr b => g b
  end
```

```
(A + B) -> (A -> C) -> (B -> C) -> C
```

```
fun (x: A + B)(f: A -> B)(g: B -> C) =>
   match x with
   | inl a => f a
   | inr b => g b
   end
```

$$\frac{\vdash A \vee B \qquad \vdash A \Rightarrow C \qquad \vdash B \Rightarrow C}{\vdash C}$$

```
empty -> A
```

```
empty -> A
```

```
fun (x: empty) => match x with end
```

```
empty -> A
```

```
fun (x: empty) => match x with end
```

$$\frac{\vdash \mathbf{False}}{\vdash A}$$

To show that a *type* **A** is inhabited, it is sufficient to construct a *program* **p : A** using *datatype constructors*, *case-analysis* and *function application*.

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To prove a *proposition* **A** it is sufficient to construct a *proof* **p** of **A** using *assumptions*, *axioms* and *derived inference rules*.

Curry-Howard correspondence

Curry-Howard correspondence

Propositions = Types

Curry-Howard correspondence

Propositions = Types

Proofs = Programs

Curry-Howard correspondence

Axioms are
datatype constructors

Curry-Howard correspondence

Axioms are
datatype constructors

Inference rules are
functions

Curry-Howard correspondence

Curry-Howard correspondence

True	`unit`

Curry-Howard correspondence

True	`unit`
False	`empty`

Curry-Howard correspondence

True	`unit`
False	`empty`
$A \wedge B$	`A × B`

Curry-Howard correspondence

True	`unit`
False	`empty`
$A \wedge B$	`A × B`
$A \vee B$	`A + B`

Curry-Howard correspondence

True	`unit`
False	`empty`
$A \wedge B$	`A × B`
$A \vee B$	`A + B`
$A \Rightarrow B$	`A → B`

Curry-Howard correspondence

Curry-Howard correspondence

modus ponens	function application

Curry-Howard correspondence

modus ponens	function application
a hypothesis	function argument

Curry-Howard correspondence

modus ponens	function application
a hypothesis	function argument
introduction rule	datatype constructor

Curry-Howard correspondence

modus ponens	function application
a hypothesis	function argument
introduction rule	datatype constructor
elimination rule	pattern matching

Programs-as-proofs are *constructive*

A program of type **A → B**,
taken as a proof, specifies how to
derive a proof of its result type **B**
from the proof of **A** *algorithmically*.

Programs-as-proofs are *constructive*

A program of type **A → B**,
taken as a proof, specifies how to
derive a proof of its result type **B**
from the proof of **A** *algorithmically*.

This is not always the case
in the *classical logic*.

Non-constructive Axioms

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- Excluded middle: $A \vee \neg A$
  - Neither a proof of A, nor of $\neg A$ is required;

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- Double negation: $((A \Rightarrow \textbf{\textit{False}}) \Rightarrow \textbf{\textit{False}}) \Rightarrow A$
  - Again, the proof of *A* is not required.

Non-constructive Axioms

- Excluded middle: $A \vee \neg A$
  - Neither a proof of A, nor of $\neg A$ is required;
- Double negation: $((A \Rightarrow \textbf{\textit{False}}) \Rightarrow \textbf{\textit{False}}) \Rightarrow A$
  - Again, the proof of A is not required.

Assuming these axioms keeps Curry-Howard correspondence *consistent* as a formal system.

What about quantifiers?

Statically-typed functional programming languages

- Haskell
- ML
- F#
- Scala
- Agda
- Coq

- Every value has a *type*
- *Type* defines a *set* of values
- Type of a program — its specification

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- Every value has a *type*
- *Type* defines a *set* of values
- Type of a program — its specification
- Types can *depend* on values

Function Type

```
A -> B
```

Function Type

```
A -> B

bool -> nat
```

Function Type

```
A -> B
```

```
bool -> nat
```

```
fun b =>
  match b with
  | true  => 0
  | false => 1
  end
```

Dependent Function Type

```
Π(x: A). B(x)
```

Dependent Function Type

```
Π(x: A). B(x)
```

```
Π(b: bool). if b then nat else unit
```

Dependent Function Type

```
Π(x: A). B(x)
```

```
Π(b: bool). if b then nat else unit
```

```
fun b =>
  match b with
  | true  => 0
  | false => tt
  end
```

Pair Type

`A × B`

Pair Type

$$A \times B$$

```
Datatype prod A B := pair of A & B
```

Dependent Pair Type

`Σ(x: A). P(x)`

Dependent Pair Type

```
Σ(x: A). P(x)
```

```
Datatype sigma A (P: A -> B):=
      ex (x: A) of P x
```

Dependent Pair Type

$$\Sigma(x: A).\ P(x)$$

```
Datatype sigma A (P: A -> B):=
      ex (x: A) of P x
```

Every value of type `sigma A P` contains
a value x of type A (*witness*)
and a value of type P(x).

Curry-Howard correspondence

Curry-Howard correspondence

$\forall x \in A, P(x)$	$\Pi(x{:}A).\ P(x)$

Curry-Howard correspondence

$\forall x \in A, P(x)$	$\Pi(x{:}A).\ P(x)$
$\exists x \in A, P(x)$	$\Sigma(x{:}A).\ P(x)$

An example of dependent type

```
Π(P : nat -> Prop).
  P(0) ->
 (Π(n : nat). P(n) -> P(n.+1)) ->
  Π(n : nat). P n
```

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```
Function nat_ind (P : nat -> Prop)
                 (f0 : P 0)
                 (fn : Π(n : nat), P n -> P (n.+1)) :=
 fun (n : nat) =>
   match n with
    | 0 => f0
    | n'.+1 => fn n' (nat_ind P f0 fn n0)
   end.
```

Dependently-typed functional programming languages

- Agda
- Coq

Proof Assistants

- Agda
- Coq

Proof Assistants

- Agda
- Coq

- *Not* Turing-complete
  - type-checking should terminate
- Proofs can be built interactively using *scripts*
- “Boring” proofs can be automated

Interactive proof construction

(in Coq)

Interactive proof construction

(in Coq)

```
Theorem counterexample (A: Type) (P: A → Prop) :
   (∃x: A, ¬P x) → ¬(∀ x, P x).
```

Interactive proof construction

(in Coq)

```
Theorem counterexample (A: Type) (P: A → Prop) :
    (∃x: A, ¬P x) → ¬(∀ x, P x).
Proof.
 case => x H1 H2.
 by apply : H1 (H2 x).
Qed.
```

Interactive proof construction

(in Coq)

```
Function counterexample :=
  fun (A : Type) (P : A -> Prop) (hyp : ∃x : A, ¬ P x) =>
  (fun F : ∀(x : A) (p : (fun x0 : A => ¬ P x0) x),
              (fun _ : (∃x0 : A, ¬ P x0) => ¬ (∀x0 : A, P x0))
                (ex_intro (fun x0 : A => ¬ P x0) x p) =>
   match hyp as e
     return ((fun _ : (∃x : A, ¬ P x) => ¬ (∀x : A, P x)) e)
   with
   | ex x x0 => F x x0
   end) (fun (x : A) (H1 : ¬ P x) (H2 : ∀x0 : A, P x0) => H1 (H2 x)).
```

Current advances

- *Four colour theorem* — mechanised in Coq in 2005
- *CompCert* — fully verified C compiler (2006)
- *Odd order theorem* — mechanised in Coq in 2013
- *Keppler's conjecture* — formally proved in 2014
- *sel4* — formally verified OS kernel (2014)

To take away

- Formal *proofs* are functional *programs* in disguise;
- *propositions* are program *types*;
- writing a proof = constructing a program;
- proving is still *tricky* and takes a mathematician’s insight;
- proof assistants help to automate the “boring” parts of mechanised formal proof constructions…
- … and the rest is a huge fun.

“A mathematician is expected to sit at his computer and think.”

Hari Seldon